离散数学（011122）

同济经管
TONGJI SEM

魏可信 kejiwei@tongji.edu.cn
https://kejiwei.github.io/

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- **4.2.1 Basic Operations of Relations**
  - Domain, range, domain (again), inverse, composition
  - Properties of basic operations
- **4.2.2 Power Operations of Relations**
  - Definition of power operations
  - Methods of power operations
  - Properties of power operations

- **Definition 4.10:** Domain, Range, and Field

  $\text{dom}R = \{x \mid \exists y\,(\langle x,y\rangle \in R)\}$

  $\text{ran}R = \{y \mid \exists x\,(\langle x,y\rangle \in R)\}$

  $\text{fld}R = \text{dom}R \cup \text{ran}R$

e.g. Example:

$R = \{\langle a,\{b\}\rangle, \langle c,d\rangle, \langle \{a\},\{d\}\rangle, \langle d,\{d\}\rangle\}$, then

$\text{dom}R = \{a, c, \{a\}, d\}$

$\text{ran}R = \{\{b\}, d, \{d\}\}$

$\text{fld}R = \{a, c, \{a\}, d, \{b\}, \{d\}\}$

- **Definition 4.11:** The *inverse* of R

$$R^{-1} = \{<y,x> \mid <x,y> \in R\}$$

- **Definition 4.12:** *Composition* of *R* and *S*

$$R \circ S = |<x,z> \mid \exists\, y\, (<x,y> \in R \wedge <y,z> \in S)\, \}$$

e.g. Example: $R=\{<1,2>, <2,3>, <1,4>, <2,2>\}$

$S=\{<1,1>, <1,3>, <2,3>, <3,2>, <3,3>\}$

$R^{-1}=\{<2,1>, <3,2>, <4,1>, <2,2>\}$

$R \circ S=\{<1,3>, <2,2>, <2,3>\}$

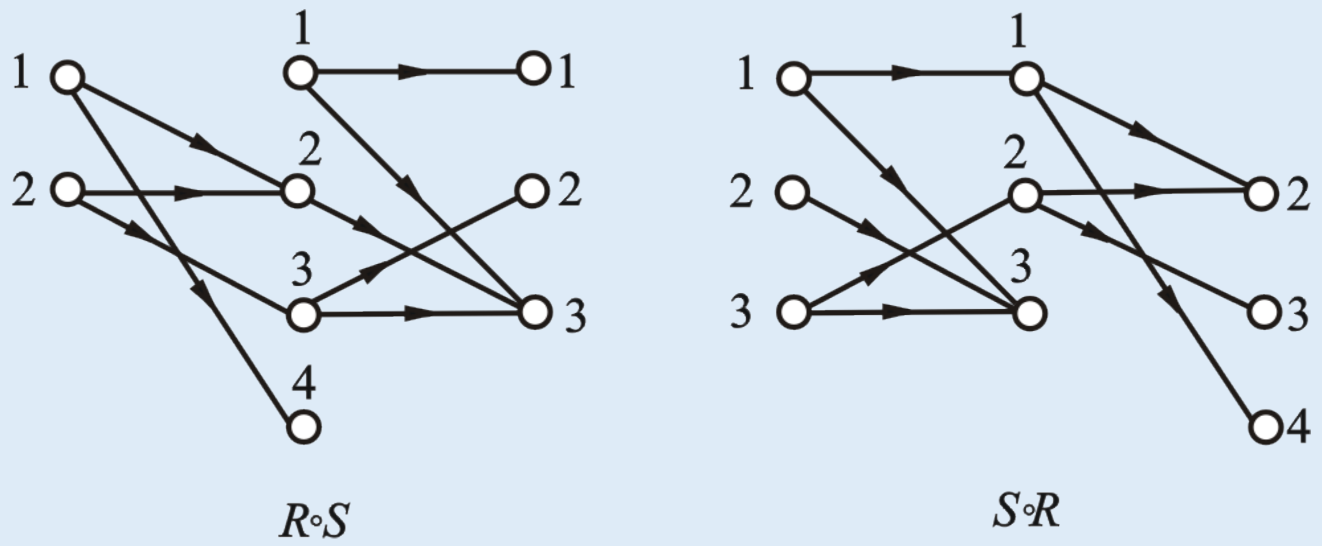
$S \circ R=\{<1,2>, <1,4>, <3,2>, <3,3>\}$

↳ Find the composition using a diagram

Use the diagrammatic (not relational diagram) method to find the composition.

e.g. **Example:** R={<1,2>, <2,3>, <1,4>, <2,2>}

S={<1,1>, <1,3>, <2,3>, <3,2>, <3,3>}



$R \circ S$ ={<1,3>, <2,2>, <2,3>}

$S \circ R$ ={<1,2>, <1,4>, <3,2>, <3,3>}

- **Theorem 4.1**: Let F be an arbitrary relation, then:
  - (1) $(F^{-1})^{-1}=F$
  - (2) $\mathrm{dom}F^{-1}=\mathrm{ran}F$, $\mathrm{ran}F^{-1}=\mathrm{dom}F$
- **Proof**:

(1) For any $<x, y>$, by the definition of inverse, we have $<x, y>\in(F^{-1})^{-1} \Leftrightarrow <y, x>\in F^{-1} \Leftrightarrow <x, y>\in F$

Thus, $(F^{-1})^{-1}=F$

(2) For any x,

$$x\in \mathrm{dom}F^{-1} \Leftrightarrow \exists y(<x, y>\in F^{-1})$$

$$\Leftrightarrow \exists y(<y, x>\in F) \Leftrightarrow x\in \mathrm{ran}F$$

Thus, $\mathrm{dom}F^{-1}= \mathrm{ran}F$.

Similarly, we can prove $\mathrm{ran}F^{-1} = \mathrm{dom}F$.

- **Theorem 4.2:**

  Let F, G, and H be arbitrary relations, then:

  (1) $(F \circ G) \circ H = F \circ (G \circ H)$

  (2) $(F \circ G)^{-1} = G^{-1} \circ F^{-1}$

- **Proof (1):** For any <x,y>,

$$<x, y> \in (F \circ G) \circ H$$
$$\Leftrightarrow \exists t\,(<x, t> \in F \circ G \wedge <t, y> \in H)$$
$$\Leftrightarrow \exists t\,(\exists s\,(<x, s> \in F \wedge <s, t> \in G) \wedge <t, y> \in H)$$
$$\Leftrightarrow \exists t\,\exists s\,(<x, s> \in F \wedge <s, t> \in G \wedge <t, y> \in H)$$
$$\Leftrightarrow \exists s\,(<x, s> \in F \wedge \exists t\,(<s, t> \in G \wedge <t, y> \in H))$$
$$\Leftrightarrow \exists s\,(<x, s> \in F \wedge <s, y> \in G \circ H)$$
$$\Leftrightarrow <x, y> \in F \circ (G \circ H)$$

Thus, $(F \circ G) \circ H = F \circ (G \circ H)$

↳ Associativity and Inverse Operations of Relational Composition

- **Theorem 4.2:**

  Let F, G, and H be arbitrary relations, then:

  (1) $(F \circ G) \circ H = F \circ (G \circ H)$

  (2) $(F \circ G)^{-1} = G^{-1} \circ F^{-1}$

- **Proof (2):** For any $\langle x, y \rangle$,

  $\langle x, y \rangle \in (F \circ G)^{-1}$

  $\Leftrightarrow \langle y, x \rangle \in F \circ G$

  $\Leftrightarrow \exists t\, (\langle y, t \rangle \in F \wedge (t, x) \in G)$

  $\Leftrightarrow \exists t\, (\langle x, t \rangle \in G^{-1} \wedge (t, y) \in F^{-1})$

  $\Leftrightarrow \langle x, y \rangle \in G^{-1} \circ F^{-1}$

  Thus $(F \circ G)^{-1} = G^{-1} \circ F^{-1}$

- **Theorem 4.3：** Composition of the Identity Relation I_A

  Let R be a relation on A, I_A is the Identity Relation on Set A

  then $R \circ I_A = I_A \circ R = R$

- **Proof For any pair $<x, y>$**

  $$<x, y> \in R \circ I_A$$
  $$\Leftrightarrow \exists t\ (<x, t> \in R \land <t, y> \in I_A)$$
  $$\Leftrightarrow \exists t\ (<x, t> \in R \land t=y \land y \in A)$$
  $$\Leftrightarrow <x, y> \in R$$

  Thus, $R \circ I_A$=R.

  Similarly, we can prove that $I_A \circ R = R$.

■ **Definition 4.13：** R^n

Let R be a relation on A, and n be a natural number. The ***n-th power of R*** is defined as:

(1) $R^0 = \{<x,x> \mid x \in A\} = I_A$

(2) $R^{n+1} = R^n \circ R$

Note:

- For any relations R_1 and R_2 on A, we have,

  $$R_1^0 = R_2^0 = I_A$$

- For any relation R on A, we have: $R^1 = R$

- For a relation R represented by a set, computing R^n means the composition of R with itself n times.
- The n-th power of a relation is equal to the n-th power of its matrix representation.
- The matrix representation of a relation is obtained by matrix multiplication, where addition is performed using logical addition.

e.g. **Example:** 设$A = \{a, b, c, d\}$, $R = \{\langle a,b\rangle,\langle b,a\rangle,\langle b,c\rangle,\langle c,d\rangle\}$,
Find the powers of R, and represent them using both a matrix and a relation diagram.

Solution: The relation matrix for R and R^2 are as follows:

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad M^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

e.g. **Example:** Let $A = \{a, b, c, d\}$, $R = \{<a,b>,<b,a>,<b,c>,<c,d>\}$, Find the powers of R, and represent them using both a matrix and a relation diagram.

Solution: The relation matrix for R^3 and R^4 are as follows:

$$M^3 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M^4 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus: $M^4 = M^2$, $R^4 = R^2$. Then we can find

$$R^2 = R^4 = R^6 = \ldots, \quad R^3 = R^5 = R^7 = \ldots$$

$R^0 = I_A$ relation matrix : $M^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

↳ Methods of power operations • relation diagram

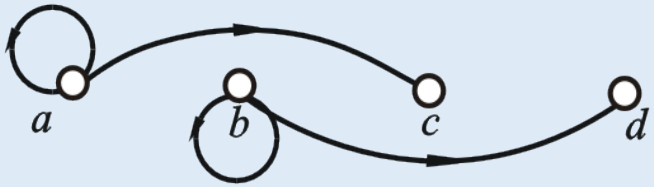
Let R be a relation on $A = \{a, b, c, d\}$,

$R = \{<a,b>,<b,a>,<b,c>,<c,d>\}$,

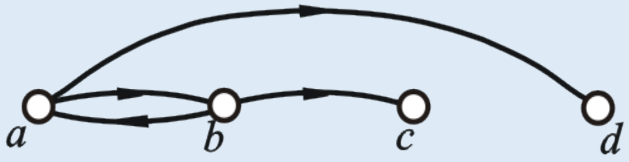
Using the relation diagram method, the relation diagrams for R^0, R^1, R^2, R^3

R^0

$R^1=R$



$R^2=R^4$



$R^3=R^5$

- **Theorem 4.4:** The periodicity or eventual stability of a power operation under finite exponents.
  Let A be a set with n elements, and let R be a relation on A.
  Then, there exist natural numbers s and t such that $R^s = R^t$.
- **Proof Outline:**

(1) A relation R on A is a subset of A×A, which contains 2^{n^2} at most pairs.

(2) Since each R^s is a subset of A×A, there are at most 2^{n^2} distinct possible relations.

(3) The sequence R: $R^0, R^1, R^2, R^3, \ldots$ has infinitely many indices but only finitely many distinct relations 2^{n^2}, so by the pigeonhole principle, there exist s≠t such that $R^s = R^t$.

- Theorem 4.5: ***Composition of Powers*** and ***Power of a Power*** properties.

  Let R be a relation on A , and $m, n \in N$, Then

  (1) $R^m \circ R^n = R^{m+n}$

  (2) $(R^m)^n = R^{mn}$

- **Proof:** By induction.

  (1) For any given $m \in N$, , we induct on n.

  If n=0, then

  $$R^m \circ R^0 = R^m \circ I_A = R^m = R^{m+0}$$

  Assume $R^m \circ R^n = R^{m+n}$, **then**

  $$R^m \circ R^{n+1} = R^m \circ (R^n \circ R) = (R^m \circ R^n) \circ R = R^{m+n+1} ,$$

  Thus, for all $m, n \in$ N 有 $R^m \circ R^n = R^{m+n}$.

- Theorem 4.5: ***Composition of Powers*** and ***Power of a Power*** properties.

  Let R be a relation on A , and $m, n \in N$, Then

  (1) $R^m \circ R^n = R^{m+n}$

  (2) $(R^m)^n = R^{mn}$

- **Proof:** By induction.

  (2) For any given $m \in N$, we induct on n.

  If $n = 0$, then

  $(R^m)^0 = I_A = R^0 = R^{m\times 0}$

  Assume $(R^m)^n = R^{mn}$, then

  $(R^m)^{n+1} = (R^m)^n \circ R^m = (R^{mn}) \circ R^m = R^{mn+m} = R^{m(n+1)}$

  Thus, for any $m, n \in N$ 有 $(R^m)^n = R^{mn}$.

- **Theorem 4.6:**, Stabilization, Periodicity and Finite State Constraint in the Powers of a Relation
  Let R be a relation on A. If there exist natural numbers s, t ($s<t$) such that $R^s = R^t$, then

(1) **For any $k \in N$, $R^{s+k} = R^{t+k}$** (*Stabilization Property*)

The two powers are equal and remain unchanged when the same power is added.

(2) **For any k, $i \in N$ $R^{s+kp+i} = R^{s+i}$, where $p = t-s$**

(*Periodicity Property*)

The period for the equality of the two powers is p

(3) **Let $S=\{R^0, R^1, \ldots, R^{t-1}\}$, then for any $q \in N$, $R^q \in S$**

(*Finite State Constraint*)

The natural number powers of a relation R on a finite set always have a period.

■ **Proof:**

(1) $R^{s+k} = R^{s} \circ R^{k} = R^{t} \circ R^{k} = R^{t+k}$

(2) Induct on k . If $k=0$, then we have

$$R^{s+0p+i} = R^{s+i}$$

Assume $R^{s+kp+i} = R^{s+i}$, where $p = t-s$, then

$$R^{s+(k+1)p+i} = R^{s+kp+i+p} = R^{s+kp+i} \circ R^{p}$$

$$= R^{s+i} \circ R^{p} = R^{s+p+i} = R^{s+t-s+i} = R^{t+i} = R^{s+i}$$

By the principle of mathematical induction, the proposition is proven.

■ Proof:

(3) For any $q \in \mathbf{N}$, if $q<t$, it is obvious that $R^q \in S$.

If $q \geq t$, then there exist natural numbers k *and* i *such that*

$$q = s+kp+i, \quad \text{where } 0 \leq i \leq p-1.$$

Thus,

$$R^q = R^{s+kp+i} = R^{s+i}$$

Since $$s+i \leq s+p-1 = s+t-s-1 = t-1$$

this proves that $R^q \in S$.

Objective :

Key Concepts :

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魏可信 kejiwei@tongji.edu.cn
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4.3.1 Definition and Determination of Relation Properties

- Reflexivity and Irreflexivity
- Symmetry and Antisymmetry
- Transitivity

4.3.2 Closure of Relations

- Definition of Closure
- Closure Calculation
- Warshall's Algorithm

- Definition 4.14: **Reflexivity and irreflexivity of Relations**

  Let R be a relation on A.

  (1) **If $\forall x(x \in A \rightarrow \langle x,x\rangle \in R)$, then R is called *reflexive* on A.**

  (2) **If $\forall x(x \in A \rightarrow \langle x,x\rangle \notin R)$, then R is called *irreflexive* on A.**

- **Reflexive:** The universal relation E_A on A , the identity relation I_A, the less-than-or-equal relation L_A, and the divisibility relation D_A.
- **Irreflexive:** The less-than relation($<$) on the real number set and the strict inclusion relation ($\subsetneq$) on the power set.

e.g.⋙ **Example:** $A = \{a, b, c\}$, R_1, R_2, R_3 is the relation on A, where

$R_1 = \{\langle a,a\rangle, \langle b,b\rangle\}$, $R_2 = \{\langle a,a\rangle, \langle b,b\rangle, \langle c,c\rangle, \langle a,b\rangle\}$, $R_3 = \{\langle a,c\rangle\}$

To determine whether the relations R_1, R_2, R_3 are reflexive or irreflexive.

- **Reflexivity check:**
  - Since (c,c)$\notin R_1$(c,c) , R_1 is **not reflexive.**
  - All of (a,a),(b,b),(c,c) are in R_2 , R_2 is **reflexive.**
  - None of (a,a),(b,b),(c,c) are present, R_3 is **not reflexive**.
- **Irreflexivity check:**
  - R_1 contains (a,a) and (b,b) ,meaning some elements have self-loops, R_1 is **not irreflexive.**
  - Since R_2 contains self-loops ((a,a),(b,b),(c,c)), R_2 is **not irreflexive**.
  - R_3 does not contain any self-loops ((x, x)), R_2 is **irreflexive**.
- **Final Summary:** R_1 : Neither reflexive nor irreflexive; R_2 : Reflexive; R_3 : Irreflexive

- Definition 4.15: **Symmetric and Antisymmetric of Relations.**

  Let *R be a relation on A*,

  (1) If $\forall x \forall y(x,y \in A \wedge \langle x,y\rangle \in R \rightarrow \langle y,x\rangle \in R)$, then R is called a *symmetric relation* on A.

  (2) If $\forall x \forall y(x,y \in A \wedge \langle x,y\rangle \in R \wedge \langle y,x\rangle \in R \rightarrow x=y)$, 则称*R* is called an *antisymmetric relation* on *A*.

- **Such as:**

  Symmetric: The universal E_A on *A*, the identity relation I_A, and the empty relation ∅.

  Antisymmetric: The identity relation I_A and the empty relation ∅ are antisymmetric relations on *A*.

Note: The formulas (1) and (2) iterates over all elements x,y in *A*, but the actual constraint applies only to the elements in ***R***.

e.g. Example 2: Let A={a,b,c}, and R1,R2,R3 and R4 are relations on A, where...

$$R_1=\{<a,a>,<b,b>\},\quad R_2=\{<a,a>,<a,b>,<b,a>\}$$

$$R_3=\{<a,b>,<a,c>\},\quad R_4=\{<a,b>,<b,a>,<a,c>\}$$

To determine whether the relations R_1, R_2, R_3 , R_4 are Symmetric or Antisymmetric.

	Symmetric check: $\forall x \forall y(x,y \in A \wedge <x,y> \in R \rightarrow <y,x> \in R)$	Antisymmetric check: $\forall x \forall y(x,y \in A \wedge <x,y> \in R \wedge <y,x> \in R \rightarrow x=y$	Conclusion
R_1	R_1 only contains (a,a) and (b,b).	Only contains reflexive elements (x,x) and does not include any (x,y) such that x≠y.	Symmetric: ✓ Yes Symmetric: ✓ Yes
R_2	Every (x,y)∈R_2, the corresponding (y,x) is also in R_2.	Contains and (b,a), but a≠b, which violates the requirement that x=y.	Symmetric: ✓ Yes Symmetric: ✗ No

↳Symmetric & Antisymmetric (e.g.)

e.g.>>>**Example:** Let A={a,b,c}, and R1,R2,R3 and R4 are relations on A, where...

$$R_1=\{<a,a>,<b,b>\}, \quad R_2=\{<a,a>,<a,b>,<b,a>\}$$

$$R_3=\{<a,b>,<a,c>\}, \quad R_4=\{<a,b>,<b,a>,<a,c>\}$$

To determine whether the relations R_1, R_2, R_3 , R_4 are Symmetric or Antisymmetric.

	Symmetric check: $\forall x \forall y(x,y \in A \wedge <x,y> \in R \rightarrow <y,x> \in R)$	**Antisymmetric check:** $\forall x \forall y(x,y \in A \wedge <x,y> \in R \wedge <y,x> \in R \rightarrow x=y$	**Conclusion**
R_3	(a,b)$\in R_3$ but (b,a)$\notin R_3$, and (a,c)$\in R_3$ but (c,a)$\notin R_3$	no pairs (x,y) and (y,x) in R_3 for x≠y, making the condition trivially true.	Symmetric: ✗ No Symmetric: ✓ Yes
R_4	(a,c)$\in R_4$ but (c,a)$\notin R_4$	(a,b)$\in R_4$ and (b,a)$\in R_4$, but a≠b	Symmetric: ✗ No Symmetric: ✗ No

↳Transitive relation

- Definition 4.16: transitive relation on A.

  Let R be a relation on A. If

  $\forall x \forall y \forall z(x,y,z \in A \wedge \langle x,y\rangle \in R \wedge \langle y,z\rangle \in R \rightarrow \langle x,z\rangle \in R)$,

  then *R is* called a ***transitive relation*** on ***A*.**

- **Such as:** The universal relation E_A on A, the identity relation I_A, the empty relation ∅, the less-than-or-equal relation, the less-than relation, the divisibility relation, the inclusion relation, and the strict inclusion relation.

e.g. Example: Let $A=\{a, b, c\}$, R_1, R_2, R_3 relation on A, where

$R_1=\{<a,a>,<b,b>\}$; $R_2=\{<a,b>,<b,c>\}$; $R_3=\{<a,c>\}$

To determine whether the relations R_1, R_2, R_3 are Transitive relation on A.

	$\forall x \forall y \forall z (x,y,z \in A \wedge <x,y> \in R \wedge <y,z> \in R \rightarrow <x,z> \in R)$	Conclusion
R_1	only contains reflexive elements and has no pairs that could violate transitivity.	Transitive relation ✓ Yes
R_2	because (a,b)$\in R_2$ and (b,c)$\in R_2$, but (a,c)$\notin R_2$	Transitive relation ✗ No
R_3	contains only a single pair and has no chains to check for transitivity violations.	Transitive relation ✓ Yes